Solution 1

1. A finite trigonometric series is of the form $a_0+\sum_{n=1}^N(a_n\cos nx+b_n\sin nx)$. A trigonometric polynomial is of the form $p(\cos x,\sin x)$ where $p(x,y)$ is a polynomial of two variables x,y. Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

   Solution Let
$$p(x,y)=\sum_{j,k,\ 1\le j+k\le N}^{N} a_{jk}x^jy^k$$
   be a polynomial of degree N. A general trigonometric polynomial is of the form
$$p(\cos x,\sin x)=\sum_{j,k} a_{jk}\cos^j x\sin^k x\ .$$
   Plugging Euler's formulas $\cos x=\frac{1}{2}(e^{ix}+e^{-ix}), \sin x=\frac{1}{2i}(e^{ix}-e^{-ix})$, into this expression, one has
$$p(\cos x,\sin x)=\sum_{j,k} a_{jk}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^j\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^k\ .$$
   Collecting the terms into series in e^{inx},
$$p(\cos x,\sin x)=\sum_{n=-N}^{N} c_ne^{inx}\ ,$$
   which is a finite Fourier series.

   Conversely, observe that $\cos 2x=\cos^2 x-\sin^2 x, \sin 2x=2\cos x\sin x$, by induction you can show that $\cos nx$ and $\sin nx$ can be expressed as $p(\cos x,\sin x)$ of degree N. Hence a finite Fourier series $f(x)=a_0+\sum_{n=1}^N(a_n\cos nx+b_n\sin nx)$ can be written as a trigonometric polynomial.

2. Let f be a 2π-periodic function which is integrable over $[-\pi,\pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx=\int_J f(x)dx,$$
   where I and J are intervals of length 2π.

   Solution It is clear that f is also integrable on $[n\pi,(n+2)\pi],\ n\in\mathbb{Z}$, so it is integrable on the finite union of such intervals. As every finite interval can be a subinterval of intervals of this type, f is integrable on any $[a,b]$. To show the integral identity it suffices to take $J=[-\pi,\pi]$ and $I=[a,a+2\pi]$ for some real number a. Since the length of I is 2π, there exists some n such that $n\pi\in I$ but $(n+2)\pi$ does not belong to the interior of I. We have
$$\int_a^{a+2\pi} f(x)dx=\int_a^{n\pi} f(x)dx+\int_{n\pi}^{a+2\pi} f(x)dx.$$
   Using
$$\int_a^{n\pi} f(x)dx=\int_{a+2\pi}^{(n+2)\pi} f(x)dx$$
   (by a change of variables), we get
$$\int_a^{a+2\pi} f(x)dx=\int_{a+2\pi}^{(n+2)\pi} f(x)dx+\int_{n\pi}^{a+2\pi} f(x)dx=\int_{n\pi}^{(n+2)\pi}\ .$$

Now, using a change of variables again we get

$$\int_{n\pi}^{(n+2)\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.$$

3. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

Solution Write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Suppose $f(x)$ is an even function. Then, for $n \geq 1$, we have

$$\pi b_n = \int_{-\pi}^{\pi} \sin nx f(x)dx = \int_{-\pi}^{0} \sin nx f(x)dx + \int_{0}^{\pi} \sin nx f(x)dx \ .$$

By a change of variable and using $f(-x) = f(x)$ since $f(x)$ is an even function,

$$\int_{-\pi}^{0} \sin nx f(x)dx = \int_{0}^{\pi} \sin(-nx) f(-x)dx = -\int_{0}^{\pi} \sin nx f(x)dx,$$

one has

$$\pi b_n = -\int_{0}^{\pi} \sin nx f(x)dx + \int_{0}^{\pi} \sin nx f(x)\, dx = 0.$$

Hence the Fourier series of every even function f is a cosine series.

Now suppose $f(x)$ is an odd function. Then, for $n \geq 1$, we have

$$\pi a_n = \int_{-\pi}^{\pi} \cos nx f(x)dx = \int_{-\pi}^{0} \cos nx f(x)dx + \int_{0}^{\pi} \cos nx f(x)dx \ .$$

By a change of variable and using $f(-x) = -f(x)$ since $f(x)$ is an odd function,

$$\int_{-\pi}^{0} \cos nx f(x)dx = \int_{0}^{\pi} \cos(-nx) f(-x)dx = -\int_{0}^{\pi} \cos nx f(x)dx,$$

one has

$$\pi a_n = -\int_{0}^{\pi} \cos nx f(x)dx + \int_{0}^{\pi} \cos nx f(x)dx = 0 \ , \quad \forall n \geq 0 \ .$$

4. Here all functions are defined on $[-\pi, \pi]$. Verify their Fourier expansion and determine their convergence and uniform convergence (if possible).

   (a)
$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)
$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

(c)

$$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

(d)

$$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

Solution

(a) Consider the function $f_1(x) = x^2$. As $f_1(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi} x^2 \sin nx \Big|_{-\pi}^{\pi} - \frac{2}{n\pi} \int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi} x \cos nx \Big|_{-\pi}^{\pi} - \frac{2}{n^2\pi} \int_{-\pi}^{\pi} \cos nx dx \\ &= 4 \frac{(-1)^n}{n^2}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = | - 4 \frac{(-1)^{n+1}}{n^2} | \leq \frac{4}{n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(b) Consider the function $f_2(x) = |x|$. As $f_2(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{2\pi} \frac{x^2}{2} \Big|_{-\pi}^{\pi} = \frac{\pi}{2},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx dx \\ &= \frac{2}{n\pi} x \sin nx \Big|_{0}^{\pi} - \frac{2}{n\pi} \int_{0}^{\pi} \sin nx dx \\ &= -\frac{2}{n^2\pi} \cos nx \Big|_{0}^{\pi} \\ &= -2 \frac{[(-1)^n - 1]}{n^2\pi}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = |2 \frac{[(-1)^n - 1]}{n^2\pi}| \leq \frac{4}{\pi n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(c) As $f(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx \\ &= \frac{2}{n\pi}\cos nx\Big|_0^{\pi} \\ &= 2\frac{[(-1)^n - 1]}{n\pi}. \end{aligned}$$

Now we consider the convergence of the series $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$. Fix $x \in (-\pi, 0) \cup (0, \pi)$, Using the elementary formula

$$\sum_{n=1}^{N}\sin(2n-1)x = \frac{\sin^2(N+1)x}{\sin x},$$

one has that the partial sums $|\sum_{n=1}^{N}\sin(2n-1)x| = |\frac{\sin^2(N+1)x}{\sin x}| \le |\frac{1}{\sin x}|$ are uniformly bounded. This also holds for $x = 0$, in which case $|\sum_{n=1}^{N}\sin(2n-1)0| = 0$. Furthermore, the coefficients $1/(2n-1)$ decreases to 0. We conclude that the Fourier series converges pointwisely by Dirichlet's test.

(d) As $g(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$. By integration by parts,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx \\ &= -\frac{2}{n\pi}x(\pi - x)\cos nx\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx dx \\ &= \frac{2}{n^2\pi}(\pi - 2x)\sin nx\Big|_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{4}{n^3\pi}\cos nx\Big|_0^{\pi} \\ &= -\frac{4}{n^3\pi}[(-1)^n - 1]. \end{aligned}$$

As

$$|b_n| \le \frac{8}{\pi n^3},$$

we conclude that the Fourier series converges uniformly by the Weierstrass M-test.

5. Let f be a π-periodic function which is infinitely many times differentiable on $\mathbb{R}$. Show that its Fourier coefficients are of order $o(1/n^k)$ for any $k \ge 1$, that is, $a_n n^k, b_n n^k \to 0$ as $n \to \infty$ for any k. Hint: Better use complex notation.

Solution. We use complex notation. Let c_n^k be the Fourier series of $f^{(k)}$. We have $c_n^k = (in)^k c_n$ for all n and k. Replacing k by $k+1$, we have

$$|c_n| \le \frac{|c_n^{k+1}|}{|(in)^{k+1}|} \le \frac{C}{n^{k+1}} ,$$

where in the second step we have applied Riemann-Lebsegue Lemma to $f^{(k+1)}$. We conclude $n^k|c_n| \le Cn^{-1} \to 0$ as $n \to \infty$.

Remark A sequence $\{a_n\}$ satisfies $a_n = \mathrm{o}(n^\sigma)$ if

$$\lim_{n\to\infty} \frac{a_n}{n^\sigma} = 0 \ .$$

It satisfies

$$a_n = \mathrm{O}(n^\sigma)$$

if there is a constant C such that

$$\frac{|a_n|}{n^\sigma} \le C \ , \quad \forall\, n \ge 1 \ .$$

6. Let f be a 2π-periodic function whose derivative exists and is integrable on $[-\pi, \pi]$. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebesgue Lemma. Hint: Use integration by parts to relate the Fourier coefficients of f to those of f'.

   Solution Performing integration by parts yields

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx dx \ .$$

   Therefore,

$$\pi |a_n| \le \frac{1}{n} \int_{-\pi}^{\pi} |f'(x)| dx \to 0 \ , \quad n \to \infty \ .$$

   Similarly the same result holds for b_n.

7. Use the previous exercise to give prove Riemann-Lebsgue Lemma. Hint: Every integrable function can be approximated by C^1-functions in appropriate sense.

   Solution For every integrable function f, given $\varepsilon > 0$, there is a step function s such that

$$\int_a^b |f - s|\, dx < \frac{\varepsilon}{2} \ .$$

   On the other hand, it is geometrically evident (by smoothly connecting the jumps) that for the step function s, given $\varepsilon > 0$, there is a C^1-function g such that

$$\int_a^b |s - g|\, dx < \frac{\varepsilon}{2} \ .$$

   The desired result follows by putting these two estimates together.

   Remark The second step can be described more analytically, but I prefer not to.

8. Let f be a continuous 2π-periodic function. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebsegue lemma. Hint: Establish the formula

$$2\pi a_n = \int_{-\pi}^{\pi} [f(y) - f(y + \pi/n)] \cos ny\, dy \ ,$$

   using Problem 2.

Solution Setting $y = z + \pi/n$, we have

$$\begin{aligned} \pi a_n &= \int_{-\pi}^{\pi} f(y)\cos ny\,dy \\ &= \int_{-\pi+\pi/n}^{\pi+\pi/n} f(z+\pi/n)\cos(z+\pi/n)\,dz \\ &= \int_{-\pi+\pi/n}^{\pi+\pi/n} f(z+\pi/n)(-\cos z)\,dz \\ &= -\int_{-\pi}^{\pi} f(y+\pi/n)\,dy\ . \end{aligned}$$

It follows that

$$2\pi a_n = \int_{-\pi}^{\pi} [f(y) - f(y+\pi/n)]\cos ny\,dy\ ,$$

and $a_n \to 0$ as $n \to \infty$. Similarly $b_n \to 0$.

9. Let f be a bounded function on $[a,b]$ and $x \in [a,b]$. f is said to be locally Lipschitz continuous at x if there are some L and δ such that

$$|f(y) - f(x)| \le L|y-x|, \quad |y-x| \le \delta,\ y \in [a,b]\ .$$

Show that f is Lipschitz continuous at x whenever it is locally Lipschitz continuous at x.

Solution For $y \in (-\delta + x, x+\delta) \cap [a,b]$ we have $|f(y) - f(x)| \le L|y-x|$ already. For $y, |y-x| > \delta$,

$$|f(y) - f(x)| \le \frac{|f(y| + f(x)|}{|y-x|}|y-x| \le \frac{2M}{\delta}|y-x|\ ,$$

where M is a bound on f. Taking $L_1 = \max\{L, 2M/\delta\}$, $|f(y) - f(x)| \le L_1|y-x|$ for all y.